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"QED: A Proof of Renormalizability"

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Preface

The central mathematical problem of quantum field theory, as it is currently formulated in terms of Euclidean Feynman integrals, is to construct a measure on the space of fields $\{\Phi(x)\}$ given by

$$d\nu(\Phi) = \text{const.}e^{\lambda V(\Phi)}dP(\Phi) \quad .$$

Here the Gaussian measure $dP(\Phi)$ is determined by the free or quadratic part $\mathcal{L}_0$ of the Lagrangian of the model; its covariance C is given by

$$\frac{1}{2}(\Phi,\ C^{-1}\Phi) = \int \mathcal{L}_0(\Phi(x))\ dx \ .$$

The potential $V(\Phi)$ is determined by the interaction part $\mathcal{L}_{int}$ of the Lagrangian:

$$V(\Phi) = -\int \mathcal{L}_{int}(\Phi(x))\ dx \ .$$

These measures may not be genuine for technical reasons - the presence of fermions or lack of regularity - and so what we ask of the above formal expression for $d\nu$ is that its moments exist.

The Gaussian measure $dP(\Phi)$ is well understood. For instance, the integral of a polynomial in Φ can be elegantly evaluated as a sum over graphs whose lines correspond to the covariance C. The non-Gaussian measure $d\nu(\Phi)$ is another matter. Postponing the question of the actual existence of $d\nu$, we can ask whether it exists in perturbation theory, i.e., as a formal power series in the coupling constant λ, again a question about Gaussian integrals. Although simpler, this question, or some version of it, has been under investigation for about half a century. The difficulties are well known: in all models of interest, the covariance $C(x,y)$ is a classical Green's function with short-distance or ultraviolet (UV) singularities as $|x-y| \to 0$; if massless fields are involved, there is in addition the long-distance or infrared (IR) problem that $C(x,y)$ does not decay exponentially as $|x-y| \to \infty$. As a result, most of the graphs in the perturbation series are infinite.

The folk remedy is to cancel these infinities by adjusting or renormalizing V with counterterms $\delta V(\Phi,\lambda)$:

$$d\nu \rightarrow d\nu_{ren} = \text{const.}e^{\lambda V+\delta V}\, dP\ .$$

The counterterms $\delta V(\Phi,\lambda)$ are permitted to have the same form as terms in the original Lagrangian, but the coefficients of these terms are formal power series in λ which themselves have infinite coefficients. The central problem of perturbative renormalization theory is to demonstrate that there is some choice of δV for which all the infinities cancel, yielding a renormalized perturbation series with finite coefficients.

Such a demonstration typically encounters severe combinatorial and graphical complexities. To each order in λ, there are many graphs. Elementary power counting considerations may indicate that a graph G is finite but such power counting is too superficial in that G may contain divergent subgraphs. So a good renormalization algorithm on a graph G must first make subtractions on the divergent subgraphs of G, beginning with the smallest. But - and this is the notorious problem of "overlapping divergences" - what if two divergent subgraphs G_1, $G_2 \subset G$ intersect and neither is a subgraph of the other? In general, the renormalization procedure on G_1 will disturb that on G_2, and vice versa. Furthermore, how can we be sure that the required subtractions on all of these graphs to all orders can be implemented by an a priori choice of δV?

It took many years and the heroic efforts of many people to chart a safe course through these difficulties. Some of the milestones in this journey were the original work on QED[1] by Feynman, Schwinger and Tomonaga, the refinements by Dyson, Matthews, Salam, ..., the Dyson-Weinberg Power Counting Theorem[2], the renormalization prescription of Bogoliubov and Parasiuk[3] and the subsequent improvements of Hepp[4] and Zimmermann[5], culminating in the 1960s in what is now known as BPHZ renormalization[6]. Still, in 1970, a student of perturbative renormalization knew that he was not embarking on a pleasure cruise.

The early 1970s brought a new set of ideas to the subject, namely the renormalization group ideas of Wilson[7]. As interpreted by Gallavotti and co-workers[8-10], this approach is based on making scale decompositions of the fields or of the covariance: $\Phi = \sum_{h=-\infty}^{\infty} \Phi^{(h)}$ or $C = \sum_{h=-\infty}^{\infty} C^{(h)}$, where $C^{(h)}$ has length scale M^{-h}, $M > 1$ being a fixed scale parameter. In effect, this decomposition resolves the UV and IR singularities of C. By successively integrating out the fields $\Phi^{(h)}$ (from high to low h), Gallavotti and Nicolò[10] obtained a natural and beautiful tree expansion for $d\nu_{ren}$. The GN tree expansion dramatically simplifies the problem of perturbative renormalization, enabling one to make a choice of counterterms δV and to renormalize scale by scale without ever seeing overlapping divergences or the usual combinatorial complexities. With the control of the GN tree expansion, it is then relatively easy to show that a renormalized graph is finite and to obtain a sharp estimate on its size.

If one wishes to apply the GN method to a gauge field model, a basic problem arises because the scale decompositions do not respect gauge invariance: the model is renormalizable but it is not clear that renormalization can be achieved using only gauge invariant counterterms δV. (For that matter, this problem arises in any renormalization scheme based on BPHZ ideas.) In this monograph we return to the original model of QED and verify that the GN method can be applied with only gauge invariant counterterms in $d\nu_{ren}$.

In this monograph we have tried hard to provide a complete exposition that will be accessible to a wide audience - not just to experts in field theory. In 1988, perturbative renormalization may still not be a pleasure cruise, but we believe that the student can look forward to a relatively easy journey which boasts a number of beautiful vistas.

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