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Computability of Julia Sets

With 31 Figures

Springer

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To our families

Preface

Among all computer-generated mathematical images, Julia sets of rational maps occupy, perhaps, the most prominent position. Their beauty and complexity can be fascinating. They also hold a deep mathematical content, and numerical experiments have become a defining feature of the subject of Complex Dynamics.

Computational hardness of Julia sets is the main subject of this book. By definition, a computable set in the plane can be visualized on a computer screen with an arbitrarily high magnification. In this definition the running time of the visualization algorithm is not limited.

Countless programs to visualize Julia sets have been written. Yet, as we will see,

it is possible to constructively produce examples of quadratic polynomials, whose Julia sets are not computable.

In a way, this result is striking – it says that while a dynamical system can be described numerically with an arbitrary precision, the picture of the dynamics cannot be visualized.

As one indication of how unusual this is, consider the following. Another interesting object for a quadratic polynomial is its filled Julia set. It is obtained by "filling in" all the holes in the Julia set itself. In doing this, the computable properties of the picture can change dramatically:

a filled Julia set is always computable.

The non-computability phenomenon is very subtle, and in describing it we will require a very precise analytic machinery. Many of the techniques we use have only become available in the last few years. Perversely, we are able to construct non-computable examples of Julia sets because we understand Julia sets so well.

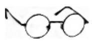
Non-computability turns out to be rare. Most Julia sets *are* computable. Their computational hardness, however, may vary. The running time required to produce a high-resolution image of a computable Julia set may be prohibitively high. Already we have seen some further surprises – a class of Julia sets (Julia sets of quadratic polynomials with parabolic orbits) empirically thought of as hard to compute turns out to be easy (and with a practical algorithm).

Our understanding of the computational complexity of Julia sets is in its first stages. Examples of a truly pathological kind (Julia sets of quadratic polynomials with Cremer periodic orbits) turn out to always be computable. No informative pictures of this type have ever been produced, as the running time of all presently existing algorithms renders them impractical. However, it is not known if they are *ever* computationally hard. This is probably the case at least sometimes, but it may also be possible that some of them can be visualized effectively by a clever algorithm. Many interesting problems await further study here.

The goal of the present book is to summarize our present knowledge about the computational properties of Julia sets in a fashion that is as self-contained as possible. While we have found the interplay between theoretical Computer Science and Dynamical Systems extremely fruitful, it makes the presentation more challenging. We have striven to make the book accessible and interesting to experts in both fields. The book assumes no prior knowledge of computability theory, and only a basic familiarity with complex analysis.

We start the book with an introduction to computability theory (Chapter 1) and a survey of the basic principles of dynamics of rational maps (Chapter 2). In Chapter 3 we begin the study of the computability and complexity of Julia sets by looking at some typical examples. We discuss the general positive results in Chapter 4. Non-computability appears in Chapter 5. Chapter 6 serves to understand the topological structure of non-computable examples in more detail.

The material we view as "optional reading" is

 typeset like this.

It is either not directly related to the main storyline, or is too technical, and is directed towards experts in one of the two fields.

Acknowledgments

It is our pleasure to thank our friend and colleague Ilia Binder for the many useful discussions on the computability of Julia sets. Some of the material presented in this book is based on our joint works. We are grateful to John Milnor for his interest in our work, and many insightful comments and questions, which have inspired much of our study. We thank our colleague Michael Shub for useful discussions of our results. Alan Baker's comments were very helpful to us in understanding the connection between number-theoretical properties of parameters and computability of Julia sets. M.B. would like to thank Stephen Cook for his guidance and for sharing his insights on Complexity Theory. Our colleague Boris Khesin was instrumental in getting this book to the publisher.

We thank the Mathematics and Computer Science Departments of the University of Toronto for the wonderful working environment. M.Y. wishes to thank IHÉS, and FIM at ETHZ, and both M.Y. and M.B. thank IPAM at UCLA, for the hospitality extended to us during our visits to these institutes while working on this book. Our research was partially supported by NSERC, through its Canada Graduate Scholarship (M.B.) and a Discovery Grant (M.Y.).

As this work was taking shape, the questions, enthusiasm, and even skepticism of our colleagues have been an invaluable motivation for us. We thank them all.

Contents

List of Notation

$B(y,r)$	the ball with center $y \in \mathbb{R}^n$ and radius r;
$B(Y,r)$	the r-neighborhood of the set Y in $\mathbb{R}^n$;
$\mathbb{U}$	the unit disk in $\mathbb{C}$;
$\mathbb{D}$	the set of dyadic rationals;
$\hat{\mathbb{C}}$	the Riemann sphere;
$\mathbb{T}$	the circle $\mathbb{R}/\mathbb{Z}$;
$\mathscr{M}$	the Mandelbrot set;
$\mathrm{Crit}(R)$	the set of critical points of a rational map R;
$\mathrm{Postcrit}(R)$	the postcritical set of R;
K_n^*	the set of all compact subsets of $\mathbb{R}^n$;
$\mathbb{R}_{\mathscr{C}}$	the set of all computable real numbers;
M^{ϕ}	an oracle Turing Machine;
$\mathbb{T}$	the circle $\mathbb{R}/\mathbb{Z}$;
S^1	the unit circle $\{\lvert z\rvert = 1\} \subset \mathbb{C}$;
$\mathscr{C}$	the set of finite unions of closed dyadic balls in $\mathbb{R}^k$;
$\mathbb{R}_{\mathscr{C}}$	the field of computable real numbers;
$\mathbb{C}_{\mathscr{C}}$	the field of computable complex numbers;
f^n	unless otherwise specified, the n-th iterate $\underbrace{f \circ f \circ \cdots \circ f}_{n}$;
$J(R)$	the Julia set of the function R;
$K(p)$	the filled Julia set of the polynomial p;
J_c	the Julia set $J(z^2+c)$;
K_c	the filled Julia set $K(z^2+c)$;
$\mathscr{B}$	the set of Brjuno numbers;
$a_n \asymp b_n$, or $a_n = \Theta(b_n)$	$a_n = O(b_n)$ and $b_n = O(a_n)$;
$a_n \lesssim b_n$	$a_n = O(b_n)$.