

Introduction to Insurance Mathematics

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Technical and Financial Features
of Risk Transfers

 Springer

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Preface

This book aims at introducing technical and financial aspects of the insurance business, with a special emphasis on the actuarial valuation of insurance products. While most of the presentation concerns life insurance, also non-life insurance is addressed, as well as pension plans.

The book has been planned assuming as target readers:

- advanced undergraduate and graduate students in Economics, Business and Finance;
- advanced undergraduate students in Mathematics and Statistics, possibly aiming at attending, after graduation, actuarial courses at a master level;
- professionals and technicians operating in insurance and pension areas, whose job may regard investments, risk analysis, financial reporting, and so on, hence implying communication with actuarial professionals and managers.

Given the assumed target, the use of complex mathematical tools has been avoided. In this sense, the book can be placed at some “midpoint” of the existing literature, part of which adopts more formal approaches to insurance problems, which implies the use of non-elementary mathematics and calculus, whereas another part addresses practical questions totally avoiding even basic mathematics (which, in our opinion, can conversely provide effective tools for presenting technical and financial features of the insurance business).

We assume that the reader has attended courses providing basic notions of Financial Mathematics (interest rates, compound interest, present values, accumulations, annuities, etc.) and Probability (probability distributions, conditional probabilities, expected value, variance, etc). As mentioned, Mathematics has been kept at a rather low level. Indeed, almost all topics are presented in a “discrete” framework, thus not requiring analytical tools like differentials, integrals, etc. Some Sections in which differential calculus has been used can be skipped, without significant losses in understanding the following material.

Some details concerning the chapters of the book can help in explaining the “rationale” underlying its structure and the choice of the materials therein included.

Chapter 1 first aims at presenting the concept of risk, focussing in particular on the (negative) consequences of some events which can concern a person, a family, a firm, and so on. Secondly, the Chapter describes the role of an insurance company, which takes individual risks, builds up a pool of risks, and bears the risk of losses caused by large numbers of events within the pool or unexpected severity of the claims.

In Chapter 2 various aspects of the risk pooling process are addressed. The effects of cross-subsidy (and, in particular, mutuality and solidarity) are illustrated. Then, referring to a simple portfolio structure, reinsurance arrangements, solvency and capital allocation are dealt with.

Hence, the first two Chapters provide the reader with an introduction to risk and insurance. Indeed, a risk-management oriented approach should underpin, in our opinion, the teaching of the insurance technique and finance. It is worth stressing that these two Chapters can fulfill the syllabus of a very short course (say, 20-25 hours) aiming to present the basics of risk identification, risk assessment, and risk management actions.

Chapters 3 to 7 focus on life insurance. Although many topics dealt with are rather traditional (life tables, discounting cash-flows, premiums and reserves for various insurance products), several issues of great current interest have been included; for example: mortality trends, best-estimate reserving, risk margins, profit assessment, linking life insurance benefits to the investment performance, unit-linked products, and so on.

Chapter 8 addresses problems related to the post-retirement income. In particular, defined contribution pension plans are addressed. The protection that an individual can obtain by underwriting appropriate benefits and financial guarantees, before and after retirement, is examined. Special emphasis is placed on life annuities as an element in post-retirement income arrangements. Risks emerging for the provider are described, with particular regard to the financial and longevity risks.

Finally, Chapter 9 deals with non-life insurance. First, an overview of the contents of non-life insurance products is provided. Then, premium calculation and related statistical bases are focused. Issues presented in Chapter 1 are progressed, in order to introduce the stochastic modeling of claim frequency, claim severity and aggregate claim amounts. An introduction to technical reserves and profit assessment concludes the Chapter.

Each chapter concludes with a section providing bibliographic references and suggestions for further reading. The list of references only includes textbooks and monographs, while disregarding papers in scientific journals, congress proceedings, research and technical reports, and so on. Our choice aims at limiting the number of citations, in line with the teaching orientation of this work.

We have successfully tested the logical structure and the contents of the book in various recent courses. In particular: a course of Insurance technique and finance for graduate students in Finance at the University of Parma; a course of Life insurance mathematics for undergraduate students in Statistics and undergraduate students in Mathematics at the University of Trieste; courses of Risk and Insurance, Life insurance technique, Non-life insurance technique and a distance-learning course of

Insurance technique for employees of a European insurance company, at the MIB School of Management in Trieste. Part of the material included in the book has been used also in CPD (Continuing Professional Development) courses of Life insurance technique for non-actuaries organized by the Italian actuarial professional body. Further, some specific topics have been delivered in short seminars and other teaching initiatives (for example: risk-management approach to insurance problems, stochastic mortality, linking life insurance benefits to the investment performance, etc).

Risks must be carefully identified, assessed and managed by all the agents (individuals, households, firms, public institutions, and so on). Risk transfer constitutes an effective tool for managing risks, and the importance of insurers in this transfer process is self-evident. Actually, the insurance business constitutes a growing market. Appropriate risk management solutions must be taken also by insurers, due to the risks they assume through their products.

If this book helps to better understand the technical and financial features of the insurance activity, the role of insurers as intermediaries in the risk pooling process and as financial intermediaries, and the basics of the risk management of an insurance business, then we have achieved our objective.

Trieste, July 2010

Annamaria Olivieri
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