Methods in Approximation

Mathematics and Its Applications

Methods in Approximation

Techniques for Mathematical Modelling

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EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

As the authors stress in their preface there are two kinds of approximations involved when doing mathematical investigations of real world phenomena. First there is the idealized mathematical model in its full complexity and then there are the approximations one must make in order to be able to do something with it. And, of course, the inaccuracies adhering to the determination of whatever measured constants are involved. Thus,

> "In every mathematical investigation the question will arise whether we can apply our results to the real world .... Consequently, the question arises of choosing those properties which are not very sensitive to small changes in the model and thus maybe viewed as properties of the real process.
>
> V.I. Arnol'd, 1978

Such are the considerations which lie at the basis of several fields in mathematics: deformation theory, perturbation theory and approximation theory. In the setting above the task of approximation theory becomes that of finding that approximate model which precisely captures those properties which are not very sensitive to small changes. This also helps to make it clear that much more is involved than neglecting small terms or doing an expansion in some ϵ or throwing away all non-linear terms. Often indeed there will not even be an obvious small quantity.

The senior author of this volume, the late Richard Bellmann, has spent much time on thinking about the why and how of approximation and has pioneered several new methods in the field. The book can be accurately described as a survey of his thoughts on the topic during the last 25 years. Much of the material dates from the middle and late seventies. Here this material is presented in a coherent fashion by Bellmann and Roth without losing the typical inventive and stimulating character of much of Bellmann's writing.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, December 1985

Michiel Hazewinkel

PREFACE

Any attempt by the applied mathematician to model a physical observation must involve approximation. Since the motivation for such modelling is to be able to predict accurately the behavior of a system under study, it becomes very important to understand the underlying approximation assumptions for any given problem. It is well understood that each problem has its own set of unique assumptions.

From a classical point of view, modelling of physical phenomenon has involved a very strict discipline of combining experimental observation with careful mathematical construction to obtain a consistent blend of experimentation and calculation which, in the end, predicts result consistent with observation. This ,very often, is the goal of applied mathematical research.

It is our contention that the search for accurate mathematical modelling is fundamentally intertwined with the ideas of approximation. We wish to clearly distinguish between two types of approximation: the physical approximation and the mathematical approximation.The modelling of any physical phenomenon involves creating a mathematical structure in which physical subunits are modelled in terms of mathematical expressions. Such expressions are approximations in the sense of characterizing certain interesting behavior. For example, in elasticity, the constitutive relations are often given as linear expressions while a more accurate plasticity model demands that nonlinear terms be added. The consequence of this type of approximation is the derivation of often a very complex set of final equations governing the behavior of the system. Since these equations may be difficult to solve , mathematical approximation may be required to obtain a solution.

The present volume is an accumulation of our thoughts on methods of modern mathematical techniques of approximations over the past several years. It is not our intention to survey the vast amount of work in this area, but rather to concentrate on selective topics in this challenging and rapidly expanding area of applied mathematical study.

Since the construction of mathematical approximation must have an underlying foundation, we devote chapter 1 to the ideas which are central to its construction. We consider the abstract vector space for it allows us to define approximation errors with precision. This,then,is our starting point.

Chapter 2 discusses polynomial approximation and we are particularly interested in studying curve fitting by segmented straight lines. These simple ideas require dynamic programming techniques for a solution. At the other end of the spectrum, we consider a three dimensional approximation which is used in the popular finite element method.

The more general ideas of polynomial splines are treated in chapter 3. Here polynomial approximation offers a very smooth representation of a function f(x) and therefore allows us to store only the coefficients of the spline if we wish to reproduce f(x), thereby saving enormous space in computer applications.

Chapter 4 discusses the ideas of quasilinearization allowing us not only to solve nonlinear differential equations in an efficient way, but also to determine approximate parameters and initial conditions of the differential equations if, somehow, the behavior of the system can be observed and measured.

Differential approximation, the subject of chapter 5, is a slightly different approximation. Here we are given a nonlinear differential equation and we seek to determine an approximate linear differential equation whose exact solution is a good approximation to the original solution in the range of interest.

Differential quadrature is considered in chapter 6. Here we return to numerical techniques to allow us to find approximate solutions to partial differential equations of the form,

$$u_t(x,t) = g(x,t,u(x,t),u_x(x,t))$$

$$u(x,0) = h(x),$$

The same approximation techniques can be used to solve higher order systems and using blends of this with quasilinearization, we can consider solving systems with partial information.

Chapter 7 discusses exponential approximation where we seek approximations of the form,

$$u(t) = \sum_{n=1}^{N} a_n e^{\lambda_n t} .$$

These techniques are used to convert the Fredholm integral equation,

$$u(t) = f(t) + \int_0^1 k(|t - s|)u(s)\,ds ,$$

into a set of differential equations representing an interesting two point boundary value problem.

Chapter 8 applies the approximation techniques to a study of the Ricatti equation which continually arises in mathematical physics.

$$u' + u^2 + p(t)\, u + g(t) = 0 \quad , \qquad u(0) = c. \tag{1}$$

Using the ideas of the maximum operation, we can characterize the solution of the Ricatti equation by means of upper and lower bounds. We can approach the solution of Eq.(1) yet another way ,via quasilinearization and take full advantage of quadratic convergence in the numerical solution.

The solution of approximate equations is discussed in chapter 9 . The idea here is to replace a difficult equation whose solution is unknown with an approximate equation whose solution is known exactly. These concepts are clearly based on classical techniques in applied mathematics as we point out by first considering perturbation methods in solving nonlinear differential equations. Here we have a precise way of defining a sequence of linear differential equations whose solutions are well known. Thru the perturbation series, the nonlinear equation is systematically approximated by a simpler set of linear equations.

Modern approximation methods such as the polynomial spline affords us a new freedom in selecting our problems and the theory of dynamic programming,for example, allows us to bring together many disciplines in defining the approximate equations themselves.

The goal of this chapter is to aquaint the reader with several of these techniques which may have immediate use in solving complex problems in applied mathematics.

Finally, chapter 10 discusses an application of the finite element method to compute the three dimensional magnetic field determination. In this example we use the polynomial approximation developed in chapter 2 to solve the problem.

In view of the computer revolution and in particular the interest in the small personal computer, solving difficult problems by approximation techniques has wide appeal. This is evident in the necessity of simplifying the mathematics thru approximation techniques. An interesting by product of the ideas on approximation is the need to develop new ways to store information. While this is discussed only briefly in the text, situations will continually arise where large storage, beyond the capacity of the computer, is required for solution.In these cases, approximation techniques can serve us very well.

The application of approximation techniques is itself a very challenging endeavor. With careful thought, the mathematician can undoubtedly develop ideas far beyond the scope of this book. We sincerely hope this is true and we look confidently into the future.

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