Generators and Relations in Groups and Geometries

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Generators and Relations in Groups and Geometries

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Introduction

Every group is represented in many ways as an epimorphic image of a free group. It seems therefore futile to search for methods involving generators and relations which can be used to detect the structure of a group.

Nevertheless, results in the indicated direction exist. The clue is to ask the right question. Classical geometry is a typical example in which the factorization of a motion into reflections or, more generally, of a collineation into central collineations, supplies valuable information on the geometric and algebraic structure. This mode of investigation has gained momentum since the end of last century. The tradition of geometric-algebraic interplay brought forward two branches of research which are documented in Parts I and II of these Proceedings.

Part II deals with the theory of reflection geometry which culminated in Bachmann's work where the geometric information is encoded in properties of the group of motions expressed by relations in the generating involutions. This approach is the backbone of the classification of motion groups for the classical unitary and orthogonal planes. The axioms in this characterization are natural and plausible. They provoke the study of consequences of subsets of axioms which also yield natural geometries whose exploration is rewarding. Bachmann's central axiom is the three reflection theorem, showing that the number of reflections needed to express a motion is of great importance. Keeping the three reflection theorem and relaxing the remaining axioms leads to geometries which can be considered a realization of Hjelmslev's ideas. On the other hand, dropping the three reflection theorem while restricting the class of groups under consideration also gives rise to geometries with motion-like automorphism groups. The statement of the three reflection theorem imposes a restriction on the number of factors in any product of generating involutions in the group of motions. This shows how important it is to know the number of factors needed to express any element in a given group.

Part I is devoted to the study of factorization in a linear group over a skewfield or a ring. Many of these groups have natural generators such as reflections in hyperplanes or transvections, and frequently the solution to the length problem is known. The minimal number of factors needed to express a transformation A is often related to the dimension of the subspace of vectors fixed under A. When the set of all involutions is a natural generating set, the result of a factorization is sometimes strikingly pretty, e.g. every orthogonal transformation is a product of two orthogonal involutions.

The situation is quite different for the exceptional groups. Among these only the groups of type G_2, which are automorphism groups of Cayley algebras, have been treated successfully. The results are included in these Proceedings.

The theory of lengths had a noticeable impact on various factorization methods of matrices used for computational purposes. This aspect is also touched on here.

Of course there are scores of situations in which a group with a distinguished set of generators plays an important role. Often they have their roots in areas other than geometry, such as algebra and topology. Part III of these Proceedings presents highlights demonstrating our claim.

One may look at generating sets containing only a few elements, in particular one can ask if there is a generating set for a group with just two or three elements of prescribed orders. Solutions to this problem for finite simple groups are given in Section III.1.

A Coxeter group is generated by involutions, and the structure of the group is given by prescribing the orders of pairs of involutions. Section III.2 contains an account of Coxeter groups and stresses the intimate connection between Coxeter groups and the geometry of buildings. Coxeter groups are among the easiest examples of groups presented by generators and relations that can be treated from an algorithmic point of view.

If the group under consideration is a Lie Group or an algebraic group, one can take advantage of the embedding of subsets into the surrounding variety. In Section III.3 this is done for conjugacy classes of the elements in an algebraic group.

Another aspect of the investigation of the structure of a group through generators is supplied by growth functions. A growth function serves to assign any group a position between abelian groups and free groups. Section III.4 gives an account of the celebrated theory revealing the connection between the curvature of a differentiable manifold and the growth of its fundamental group. In our opinion this representation is also accessible to the nonspecialist.

In the case just mentioned the group is finitely generated, thus the growth function is given by counting the number of different products of a certain length. Dealing with similar questions in a compact group, the Haar measure takes the place of the counting measure. Our Proceedings contain two contributions treating these topics. Section III.5 comprises a detailed description of the characterization of p-adic analytic groups in the class of pro p-groups as given by Lubotzky and Mann.

Prominent among the profinite groups are the Galois groups of infinite algebraic field extensions. It is well known that free profinite groups occur in abundance as subgroups of these groups. Section III.6 gives a very general theorem illustrating this for the case of countable Hilbertian fields. In the absolute Galois group of such a field the subgroup generated by finitely many extensions of the base field, which are either Henselian or real closed, and of finitely many elements $\tau_1, \ldots, \tau_n$ has with probability one the following structure, it is the free product of the given Galois groups and the free profinite group on $\tau_1, \ldots, \tau_n$.

Free products of finite groups and free groups are typical examples for discontinuous groups acting on manifolds in low dimensions. The final section of these Proceedings is devoted to presentations of these groups. So we give some insight into the methods of combinatorial group theory which deals with generators and relations in a manner somewhat different from the methods used in the previous part of the Proceedings.

We hope that these Proceedings can persuade the reader to see a unity of purpose in the cross section of mathematics chosen by us, although at first glance one might think that the topics presented here are quite diverse.

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